Bessel F-isocrystals for reductive groups

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Based on joint works with Xinwen Zhu/ with Masoud Kamgarpour, Lingfei Yi

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Outline

- Part I: Kloosterman sum and its generalizations
- Part II: Bessel connection and hypergeometric connection
- Part III: Frobenius structure on connection

Exponential sums

Exponential sums are any type of finite sums of complex numbers

$$S = \sum_{n=1}^{N} \exp(2\pi i \theta_n), \qquad \theta_n \in \mathbb{R}.$$

They play an important role in number theory.

Question

What is the value of S / magnitude of $|S|$?

Trivial one: $|S| \leq N$.

Kloosterman sum

The Kloosterman sum is defined for an integer $n \geq 2$, a prime p and $a \in \mathbb{F}_p^\times$ by

$$\mathrm{Kl}(n,a) = \sum_{x_i \in \mathbb{F}_p^\times} \exp\left(\frac{2\pi i}{p}\left(x_1 + x_2 + \cdots + x_{n-1} + \frac{a}{x_1 \cdots x_{n-1}}\right)\right).$$

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- $\mathrm{Kl}(2, a)$ first appeared in Fourier expansion of Poincaré series (Poincaré, 1912).
- Kloosterman (1924) obtained a (rough) estimate

$$|\,\mathrm{Kl}(2, a)| \leq 2p^{3/4}.$$

- Further estimation are studied by Carlitz, Salié, Weil and etc.

Weil bound and equidistribution law

- The best estimate (called *Weil bound*) was obtained by Weil (n=2, 1948) and Deligne (1977):

$$|\,\mathrm{Kl}(n,a)| \leq np^{(n-1)/2}.$$

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- *Equidistribution law* (Deligne and Katz).
  For example ($n=2$), one can define an angle $\theta(a) \in [0,\pi]$:

$$2p^{1/2}\cos(\theta(a)) = -\,\mathrm{Kl}(2,a) \quad \in \mathbb{R}\cap\overline{\mathbb{Q}}.$$

  Then:

$$\lim_{p\to+\infty} \frac{\sharp\{a\in\mathbb{F}_p^{\times}, \alpha\leq\theta(a)\leq\beta\}}{p-1} = \frac{2}{\pi}\int_{\alpha}^{\beta}\sin^2\theta d\theta.$$

Theorem (Deligne, SGA 4.5 (1977))

There exists an ℓ-adic local system Kl_n of rank n on $\mathbb{G}_{m,\mathbb{F}_p}$, called Kloosterman sheaf, such that

(1) *For any closed point $a \in \mathbb{G}_m(\mathbb{F}_p) = \mathbb{F}_p^\times$,*

$$\mathrm{Tr}(\mathrm{Frob}_a, \mathrm{Kl}_{n,\bar{a}}) = \mathrm{Kl}(n, a).$$

(2) Kl_n *is pure of weight $n-1$ (i.e. each Frobenius eigenvalue at each closed point of $\mathbb{G}_m(\mathbb{F}_p)$ has absolute value $p^{\frac{n-1}{2}}$).*

The Weil bound $|\mathrm{Kl}(n, a)| \leq np^{\frac{n-1}{2}}$ follows from (1-2).

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Katz (1990) calculated its monodromy group (which implies the equidistribution law)

$$G_{\mathrm{geo}}(\mathrm{Kl}_n) = \begin{cases} \mathrm{Sp}_n & n \text{ even}, \\ \mathrm{SL}_n & p, n \text{ odd}, \\ \mathrm{SO}_n & p = 2, n \text{ odd}, n \neq 7, \\ G_2 & p = 2, n = 7. \end{cases}$$

Hypergeometric sums/sheaves, after Katz

- $\psi = \exp(\frac{2\pi i}{p} -) : \mathbb{F}_p \to \mathbb{C}^\times$ additive characters;
- $n \geq m$ two integers, $\underline{\chi} = (\chi_1, \ldots, \chi_n)$, $\underline{\rho} = (\rho_1, \ldots, \rho_m)$ two paris of multiplicative characters $\mathbb{F}_p^\times \to \mathbb{C}^\times$.

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- For $a \in \mathbb{F}_p^\times$, hypergeometric sum $\mathrm{Hyp}^{(n,m)}(\psi; \underline{\chi}; \underline{\rho})(a) =$

$$\sum_{x_1 \cdots x_n = a y_1 \cdots y_m} \psi\left(\sum_{i=1}^n x_i - \sum_{i=1}^m y_i\right) \prod_{i=1}^n \chi_i(x_i) \prod_{i=1}^m \rho_i\left(y_i^{-1}\right).$$

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  When $\chi_i = 1$ are trivial, $m = 0$, it recovers Kloosterman sum.
- Katz defined its sheaf-theoretic incarnation, obtained the Weil bound: if χ_i, ρ_j are non-isomorphic $\forall i, j$, then

$$|\,\mathrm{Hyp}^{(n,m)}(\psi; \underline{\chi}; \underline{\rho})(a)| \leq n p^{(n+m-1)/2},$$

  and also the equidistribution law in many cases.

Heinloth-Ngô-Yun's Kloosterman sheaves for reductive groups

- Heinloth-Ngô-Yun reinterpreted the construction of Kl_n in the context of (geometric) Langlands program over the function field $\mathbb{F}_p(t)$ (for GL_n) and generalised it for reductive groups.

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- Heinloth-Ngô-Yun reinterpreted the construction of Kl_n in the context of (geometric) Langlands program over the function field $\mathbb{F}_p(t)$ (for GL_n) and generalised it for reductive groups.
- Let F be a global function field $\mathbb{F}_p(X)$ of a smooth curve X. The Langlands program relates
  - Automorphic forms: e.g. $f : \mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A}_F) \to \overline{\mathbb{Q}}_\ell$.
  - Galois representations: e.g. $\rho : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$. (Regard it as: ℓ-adic local system on an open subset U of X).
- The Langlands program involves reductive groups. Let G be a reductive group over $\mathbb{F}_p$ and $\check{G}$ the Langlands dual group of G. e.g.

$$G = \mathrm{GL}_n, \mathrm{SO}_{2n+1}, \mathrm{SO}_{2n}, \qquad \check{G} = \mathrm{GL}_n, \mathrm{Sp}_{2n}, \mathrm{SO}_{2n}\,.$$

- Heinloth-Ngô-Yun explicitly constructed an automorphic form f (Hecke eigenform) on G over $\mathbb{F}_p(t)$, which is
  - unramified on $\mathbb{G}_m$;
  - Steinberg representation at 0;
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- (i) Geometrize f as an automorphic sheaf Aut_f on the moduli stack of G-bundles on $\mathbb{P}^1$ with certain level structures;
  (ii) Define *the Kloosterman sheaf* $\mathrm{Kl}_{\check{G}}$ *of* $\check{G}$ as the Langlands parameter associated to Aut_f.
  It is an ℓ-adic $\check{G}$-local system on $\mathbb{G}_m$:

$$\begin{aligned} \mathrm{Kl}_{\check{G}} : \mathbf{Rep}(\check{G}) &\to \mathrm{LocSysm}(\mathbb{G}_{m,\mathbb{F}_p}), \qquad \mathrm{Kl}_{\check{G}} : \pi_1(\mathbb{G}_{m,\mathbb{F}_p}) \to \check{G}(\overline{\mathbb{Q}}_\ell) \\ V &\mapsto \mathrm{Kl}_{\check{G},V}. \end{aligned}$$

When $\check{G} = \mathrm{GL}_n, \mathrm{SL}_n$, $\mathrm{Kl}_{\mathrm{GL}_n,\mathrm{Std}} = \mathrm{Kl}_{\mathrm{SL}_n,\mathrm{Std}} = \mathrm{Kl}_n$.

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- Taking Frobenius traces, one obtain exponential sums

$$\mathrm{Kl}_{\check{G},V}(-) : \mathbb{F}_p^\times \to \mathbb{C}$$

  satisfying certain Weil bound and equidistribution law.

Explicit exponential sums

For $a \in \mathbb{F}_p^\times$

(i) (Kloosterman sum) $\mathrm{Kl}_{\mathrm{GL}_n,\mathrm{Std}}(a) = \mathrm{Kl}_{\mathrm{SL}_n,\mathrm{Std}}(a) = \mathrm{Kl}(n,a)$

$$= \sum_{x_i \in \mathbb{F}_p^\times} \exp\left(\frac{2\pi i}{p}\left(x_1 + x_2 + \cdots + x_{n-1} + \frac{a}{x_1 \cdots x_{n-1}}\right)\right),$$

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(iii) $\mathrm{Kl}_{\mathrm{SO}_3,\mathrm{Std}}(a) = \mathrm{Kl}(2,a)^2 - p = \mathrm{Kl}_{\mathrm{SL}_2,\mathrm{Sym}^2}(a)$,

(iv) $\mathrm{Kl}_{\mathrm{SO}_{2n+1},\mathrm{Std}}(a) = \sum_{x,y \in \mathbb{F}_p^\times, xy=a} \mathrm{Kl}_{\mathrm{SO}_3,\mathrm{Std}}(x)\,\mathrm{Kl}(2n-2,y)$,

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(v) $\mathrm{Kl}_{\mathrm{SO}_{2n+2},\mathrm{Std}}(a) =$

$$\sum_{x_i \in \mathbb{F}_p^\times} \exp\Big(\frac{2\pi i}{p}(x_1 + x_2 + \cdots + x_{2n} + a\frac{x_1 + x_2}{x_1 x_2 \cdots x_{2n}})\Big) + p^n - p^{n-1}$$

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Weil bounds:

$|\,\mathrm{Kl}_{\mathrm{SO}_{2n+1},\mathrm{Std}}\,| \leq (2n+1)p^n$, $|\,\mathrm{Kl}_{\mathrm{SO}_{2n+2},\mathrm{Std}}\,| \leq (2n+2)p^n$.

Questions

(i) What is the (geometric) monodromy group $G_{\mathrm{geo}}(\mathrm{Kl}_{\check{G}})$ of $\mathrm{Kl}_{\check{G}}$? (What is its equidistribution law?)

- $G_{\mathrm{geo}}(\mathrm{Kl}_{\check{G}}) :=$ Zariski closure of $\mathrm{Kl}_{\check{G}} : \pi_1^{\text{ét}}(\mathbb{G}_{m,\overline{\mathbb{F}}_p}) \to \check{G}(\overline{\mathbb{Q}}_\ell)$.

(ii) (Conjecture of Heinloth-Ngô-Yun).
Functorial properties of $\mathrm{Kl}_{\check{G}}$.
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(iii) Generalizations of hypergeometric sheaves for reductive groups.

With Masoud Kamgarpour, Lingfei Yi, we obtain generalizations for classical groups using geometric Langlands correspondence.

Theorem

If $\check{G}$ is almost simple, then the monodromy groups of $\mathrm{Kl}_{\check{G}}$ are connected and of following type:

$\check{G}$	$G_{\mathrm{geo}}(\mathrm{Kl}_{\check{G}}) \hookrightarrow \check{G}$
$A_{2n}(p > 2)$	A_{2n}
A_{2n-1}, C_n	C_n
$A_{2n}(p = 2, n \neq 3), B_n, D_{n+1}(n \geq 4)$	B_n
E_7	E_7
E_8	E_8
E_6, F_4	F_4
$A_6(p = 2), B_3, D_4, G_2$	G_2

Type A: Katz;
HNY obtain this result except some small characteristic cases;
With Xinwen Zhu, we provide a new proof from the p-adic aspect.

Theorem (Functoriality conjecture, X.–Zhu)

If $\check{H} \subset \check{G}$ in the same line of the above table, one can identify $\mathrm{Kl}_{\check{G}}, \mathrm{Kl}_{\check{H}}$ by pushout, i.e.

$$\mathrm{Kl}_{\check{G}} = \mathrm{Kl}_{\check{H}} \circ \iota : \mathbf{Rep}(\check{G}) \to \mathbf{Rep}(\check{H}) \to \mathrm{LocSysm}(\mathbb{G}_{m,\mathbb{F}_p}).$$

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This allows us to identify different exponential sums:

- $\mathrm{Kl}_{\mathrm{SO}_{2n+2},\mathrm{Std}}(a) - p^n = \mathrm{Kl}_{\mathrm{SO}_{2n+1},\mathrm{Std}}(a)$,
- if $p = 2$, $\mathrm{Kl}_{\mathrm{SO}_{2n+1},\mathrm{Std}}(a) = \mathrm{Kl}(2n+1, a)$,
- if $p > 2$, $\mathrm{Kl}_{\mathrm{SO}_{2n+1},\mathrm{Std}}(a)$ is equal to $\mathrm{Hyp}^{(2n+1,1)}(\psi; \underline{1}; \rho) =$

$$\frac{\sum_{x_i \in \mathbb{F}_p^\times} \exp\left(\frac{2\pi i}{p}\left(x_1 + x_2 + \cdots + x_{2n+1} - \frac{x_1 \ldots x_{2n+1}}{4a}\right)\right)\rho\left(\frac{x_1 \ldots x_{2n+1}}{4a}\right)}{G(\rho)}$$

Here ρ is the quadratic character, $G(\rho)$ Gauss sum.

Our proof studies *Frobenius structure* on differential equaitions.

Bessel differential equation

- The classical Bessel differential equation with a parameter λ

$$\mathrm{Be}_2 : \left(x\frac{d}{dx}\right)^2(f) - \lambda^2 x \cdot f = 0$$

admits a unique holomorphic solution on $\mathbb{C}$:

$$\frac{1}{2\pi i}\int_{S^1} \exp\lambda\left(z+\frac{x}{z}\right)\frac{dz}{z} = \sum_{r\geq 0}\frac{\lambda^{2r}}{(r!)^2}x^r.$$

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- This integration can be viewed as a continuous analogue of the discrete Kloosterman sums

$$\mathrm{Kl}(2, a) = \sum_{z\in\mathbb{F}_p^\times} \exp\left(\frac{2\pi i}{p}(z + \frac{a}{z})\right).$$

Bessel connection for reductive groups

- K a field of characteristic 0,
  $\check{G}$ a split reductive group of rank r, $\check{T} \subset \check{B} \subset \check{G}$,
  $\check{\mathfrak{g}}$ its Lie algebra, h its Coxeter number.

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- Frenkel and Gross define a $\check{\mathfrak{g}}$-valued connection on the trivial $\check{G}$-bundle on $\mathbb{G}_{m,K} = \mathrm{Spec}(K[x, x^{-1}])$ by

$$\mathrm{Be}_{\check{G}} = d + (p_{-1} + \lambda^h x p_r)\frac{dx}{x}.$$

$p_{-1} = \sum_{\text{simple root } \alpha_i} X_{-\alpha_i}$, $X_{-\alpha_i}$ a basis vector in $\check{\mathfrak{g}}_{-\alpha_i}$.
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  p_r a basis vector in $\check{\mathfrak{g}}_{\theta}$, θ maximal root.
- It has regular singularity at 0, irregular singularity at ∞.

$$\begin{aligned} \mathrm{Be}_{\check{G}} : \mathbf{Rep}(\check{G}) &\to \mathrm{Conn}(\mathbb{G}_m), \\ \rho : \check{G} \to \mathrm{GL}(V) &\mapsto \mathrm{Be}_{G,V} : d + d\rho(p_{-1} + \lambda^h x p_r)\frac{dx}{x}. \end{aligned}$$

- $\check{G} = \mathrm{GL}_n$, $V = \mathrm{Std}$, $\mathrm{Be}_{\mathrm{GL}_n,\mathrm{Std}} = d + \begin{pmatrix} 0 & \dots & 0 & \lambda^n x \\ 1 & \ddots & & 0 \\ \vdots & \ddots & 0 & \vdots \\ 0 & \dots & 1 & 0 \end{pmatrix} \frac{dx}{x}$.

$$\rightsquigarrow \left(x\frac{d}{dx}\right)^n (f) - \lambda^n x \cdot f = 0.$$

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- $\check{G} = \mathrm{SO}_{2n+1}$, $V = \mathrm{Std}$,

$$\mathrm{Be}_{\mathrm{SO}_{2m+1},\mathrm{Std}} = d + \begin{pmatrix} 0 & 0 & \dots & 2\lambda^{2n}x & 0 \\ 1 & 0 & & & 2\lambda^{2n}x \\ 0 & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix} \frac{dx}{x}.$$

$$\rightsquigarrow \left(x\frac{d}{dx}\right)^{2n+1} (f) - \lambda^{2n} x \left(4x\frac{d}{dx} + 2\right) \cdot f = 0.$$

Hypergeometric connection for classical groups

- $\check{G}$ a classical group of rank r: SL_{r+1}, SO_{2r+1}, Sp_{2r}, SO_{2r+2}, $\check{\mathfrak{g}} = \mathfrak{n}^- \oplus \mathfrak{t} \oplus \mathfrak{n}$, such that $p_{-1} \in \mathfrak{n}^-$.
- $2\check{\rho}$ = sum of all positive roots.
- $\{p_{-1}, 2\check{\rho}, p_1\}$ a principal $\mathfrak{sl}_2 \subset \check{\mathfrak{g}}$, $p_1 \in \mathfrak{n}^{p_1} \subset \mathfrak{n}$.

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- $\{p_{-1}, 2\check{\rho}, p_1\}$ a principal $\mathfrak{sl}_2 \subset \check{\mathfrak{g}}$, $p_1 \in \mathfrak{n}^{p_1} \subset \mathfrak{n}$.
- $\{d_1, \ldots, d_r\}$ degrees of fundamental invariant of $\check{\mathfrak{g}}$.
- $\{p_1, \ldots, p_r\}$ a homogeneous basis $\mathfrak{n}^{p_1}$ with $\deg(p_i) = d_i - 1$.
- Fix a fundamental degree $d > \frac{h}{2}$, we consider $\check{G}$-connection:

$$\mathrm{Hyp}_{\check{G}}(\underline{\lambda}) = d + (p_{-1} + \sum_{d_i \geq d} \lambda_i p_i x)\frac{dx}{x}, \quad \lambda_i \in K.$$

Hypergeometric connection for classical groups

For $\check{G} = \mathrm{SL}_{r+1}$, fundamental degrees $\{2, 3, \ldots, r+1\}$.
We take p_{-1}, p_1 as follows and $p_k = p_1^k$, $k = 1, \ldots, r$.

$$p_{-1} = \begin{pmatrix} 0 & \ldots & & 0 & 0 \\ 1 & \ddots & & & 0 \\ 0 & 1 & \ddots & & \vdots \\ \vdots & & \ddots & 0 & \vdots \\ 0 & & \ldots & 1 & 0 \end{pmatrix}, \quad p_1 = \begin{pmatrix} 0 & r & \ldots & 0 & 0 \\ \vdots & 0 & 2(r-1) & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & & 0 & r \\ 0 & 0 & \ldots & \ldots & 0 \end{pmatrix}$$

Hypergeometric connection $\mathrm{Hyp}_{\check{G}}(\underline{\lambda})_{\mathrm{Std}} \rightsquigarrow$ hypergeometric differential equation (Katz):

$$\left(x\frac{d}{dx}\right)^n(f) - x\left(\sum_{i=0}^{m} \mu_i (x\frac{d}{dx})^i\right)(f) = 0, \quad \mu_i \in K.$$

Be_2 vs Kl_2

- Kl_2: $\mathscr{L}_\psi$: Artin-Scheier sheaf $\pi_1(\mathbb{A}^1_{\mathbb{F}_p}) \to \mathbb{F}_p \xrightarrow{\psi} \overline{\mathbb{Q}}_\ell^\times$.

$$\mathbb{G}_m \xleftarrow{\ \pi\ } \mathbb{G}_m^2 \xrightarrow{\ \sigma\ } \mathbb{A}^1,$$

$$\sigma : (x, y) \mapsto x + y$$
$$\pi : (x, y) \mapsto xy$$

$$\mathrm{Kl}_2 := \mathrm{R}^1\, \pi_!(\sigma^*(\mathscr{L}_\psi)) \xrightarrow{\sim} \mathrm{R}^1\, \pi_*(\sigma^*(\mathscr{L}_\psi))$$

Be_2 vs Kl_2

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$$\begin{array}{ccccc} & & \mathbb{G}_m^2 & & \\ & \swarrow^{\pi} & & \searrow^{\sigma} & \\ \mathbb{G}_m & & & & \mathbb{A}^1, \end{array} \qquad \begin{aligned} \sigma &: (x,y) \mapsto x+y \\ \pi &: (x,y) \mapsto xy \end{aligned}$$

$$\mathrm{Kl}_2 := \mathrm{R}^1\,\pi_!(\sigma^*(\mathscr{L}_\psi)) \xrightarrow{\sim} \mathrm{R}^1\,\pi_*(\sigma^*(\mathscr{L}_\psi))$$

- Let K be a field of characteristic zero. Exponential $\mathscr{D}$-module $e^{\lambda x} = (\mathscr{O}_{\mathbb{A}^1}, \nabla = d - \lambda dx)$ on $\mathbb{A}^1_K$, where λ is a parameter in K, is an analogue of Artin-Scheier sheaf.
- The Bessel equation: connection on $\mathbb{G}_{m,K}$:

$$\mathrm{Be}_2 = d + \begin{pmatrix} 0 & \lambda^2 x \\ 1 & 0 \end{pmatrix} \frac{dx}{x}.$$

Then we have $\mathrm{R}^1\,\pi_!(\sigma^*(e^{\lambda x})) \simeq \mathrm{Be}_2$ as algebraic $\mathscr{D}$-modules.

Frobenius structure on Be_2

- (Dwork) $K = \mathbb{Q}_p(\lambda)$. Frobenius pullback by $x \mapsto x^p$ on $\mathbb{A}^1_K$,

$$e^{\lambda x} = d - \lambda dx \mapsto d - p\lambda x^{p-1} dx.$$

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- $\exists$ "Frobenius structure" on $e^{\lambda x}$: $F_\lambda(x)=e^{\lambda(x^p-x)}\in A^\dagger$ s.t.

$$\frac{dF_\lambda}{dx}F_\lambda(x)^{-1}+\lambda=p\lambda x^{p-1},\quad \text{i.e. } F_\lambda: F^*(e^{\lambda x})\xrightarrow{\sim} e^{\lambda x}.$$

$$\begin{aligned} A^\dagger &= \text{ring of } p\text{-adic analytic functions on a} \\ &\quad \text{closed disc of radius} > 1 \ (p\text{-adic topology}), \\ &= \bigcup_{r>1}\Big(K\langle\frac{t}{r}\rangle=\{\sum_{i\geq 0}a_i(\frac{t}{r})^i \,|\, |a_i|_p\to 0\}\Big). \end{aligned}$$

- A choice of $\lambda, \lambda^{p-1}=-p\leftrightarrow \psi_\lambda:\mathbb{F}_p\to K^\times$ s.t. for $a\in\mathbb{F}_p$, a Teichmuller lifting $[a]\in\mathbb{Z}_p$

$$F_\lambda([a])=\psi_\lambda(a).$$

$K = \mathbb{Q}_p(\lambda)$ with $\lambda^{p-1} = -p$ s.t. $\psi_\lambda = \exp(\frac{2\pi i}{p} -)$ via $K \to \mathbb{C}$.

Theorem (Dwork)

(i) *There exists a unique* $\varphi(x) \in \mathrm{GL}_2(A^\dagger)$ *satisfying*

$$x\frac{\partial \varphi}{\partial x}\varphi^{-1} + \varphi \begin{pmatrix} 0 & \lambda^2 x \\ 1 & 0 \end{pmatrix} \varphi^{-1} = p \begin{pmatrix} 0 & \lambda^2 x^p \\ 1 & 0 \end{pmatrix}$$

That is, a horizontal isomorphism $\varphi : F^*(\mathrm{Be}_2) \xrightarrow{\sim} \mathrm{Be}_2$*.*
(ii) *For every* $a \in \mathbb{F}_p^\times$*, we have*

$$\mathrm{Tr}\,\varphi([a]) = \mathrm{Kl}(2, a).$$

(iii) *For every* $a \in \mathbb{F}_p^\times$*, the p-adic absolute values of two eigenvalues of* $\varphi([a])$ *are* $|\alpha|_p = 1$ *and* $|\beta|_p = p^{-1}$*. (Behaves like an ordinary elliptic curve* $/\mathbb{F}_p$*.)*

Frobenius structure on Bessel connection

Theorem (X.–Zhu)

$K = \mathbb{Q}_p(\lambda)$ with $\lambda^{p-1} = -p$.

(i) *There exists a unique $\varphi(x) \in \check{G}(A^\dagger)$ satisfying*

$$x\frac{\partial \varphi}{\partial x}\varphi^{-1} + \mathrm{Ad}_\varphi(p_{-1} + \lambda^h x p_r) = p(p_{-1} + \lambda^h x^p p_r),$$

i.e. φ defines a "Frobenius structure" on $\mathrm{Be}_{\check{G}}$.

(ii) *For every $a \in \mathbb{F}_p^\times$ and $V \in \mathbf{Rep}(\check{G})$*

$$\mathrm{Tr}(\varphi([a]), \mathrm{Be}_{\check{G},V}) = \mathrm{Tr}(\mathrm{Frob}_a, (\mathrm{Kl}_{\check{G},V})_{\bar{a}}).$$

(iii) *When $\check{G}$ is classical or G_2, the p-adic absolute values of eigenvalues of $\varphi([a]) \in \check{G}(K)$ are same as those of $\check{\rho}(p)$, where $\check{\rho} : \mathbb{G}_m \to \check{T}$ is the half sum of positive roots.*

$(\mathrm{Be}_{\check{G}}, \varphi)$ forms a p-adic $\check{G}$-local system $\mathrm{Be}^{\dagger}_{\check{G}}$ on $\mathbb{G}_{m,\mathbb{F}_p}$.
Based on the calculation of $G_{\mathrm{diff}}(\mathrm{Be}_{\check{G}})$ (Frenkel-Gross).

Theorem

If $\check{G}$ is almost simple, then the geometric monodromy group of $\mathrm{Be}^{\dagger}_{\check{G}}$ is connected and of following type:

$\check{G}$	$G_{\mathrm{geo}}(\mathrm{Be}^{\dagger}_{\check{G}}) \hookrightarrow \check{G}$
$A_{2n}(p > 2)$	A_{2n}
A_{2n-1}, C_n	C_n
$A_{2n}(p = 2, n \neq 3), B_n, D_{n+1}(n \geq 4)$	B_n
E_7	E_7
E_8	E_8
E_6, F_4	F_4
$A_6(p = 2), B_3, D_4, G_2$	G_2

Recover the calculation of $G_{\mathrm{geo}}(\mathrm{Kl}_{\check{G}})$ due to Katz and HNY.

Theorem (Functoriality)

If $\check{H} \subset \check{G}$ in the same line of the above table, one can identify $\mathrm{Be}^{\dagger}_{\check{G}}, \mathrm{Be}^{\dagger}_{\check{H}}$ (and hence $\mathrm{Kl}_{\check{G}}, \mathrm{Kl}_{\check{H}}$) by pushout, i.e.

$$\mathrm{Kl}_{\check{G}} = \mathrm{Kl}_{\check{H}} \circ \iota : \mathbf{Rep}(\check{G}) \to \mathbf{Rep}(\check{H}) \to \mathrm{LocSysm}(\mathbb{G}_{m,\mathbb{F}_p}).$$

Such a relationship for connections $\mathrm{Be}_{\check{G}}, \mathrm{Be}_{\check{H}}$ follows from their definition.
Then the assertion follows from the uniqueness of Frobenius structure on $\mathrm{Be}_{\check{G}}$.

Proof of main theorem

- If we apply HNY's construction in the de Rham/ p-adic setting, we obtain a $\check{G}$-local system $\mathrm{Kl}_{\check{G}}^{\mathrm{dR}}$/ $\mathrm{Kl}_{\check{G}}^{\mathrm{rig}}$.
- Geometric Satake equivalence for arithmetic $\mathscr{D}$-modules.

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$$\mathrm{Be}_{\check{G}} \simeq \mathrm{Kl}_{\check{G}}^{\mathrm{dR}}.$$

- For certain algebraic connection, show its (relative) rigid cohomology is isomorphic to its (relative) algebraic de Rham cohomology. $\exists$ an isomorphism of arithmetic $\mathscr{D}$-modules.

$$\left(\mathrm{Be}_{\check{G}}\right)^{\mathrm{an}} \xrightarrow{\sim} \mathrm{Kl}_{\check{G}}^{\mathrm{rig}}.$$

- The Frobenius structure on $\mathrm{Kl}_{\check{G}}^{\mathrm{rig}}$ gives rise to a Frobenius structure on $\mathrm{Be}_{\check{G}}$.

Hypergeometric sheaves for classical groups

Theorem (Kamgarpour–X.–L. Yi)

(i) *There exists an automorphic function (resp. automorphic sheaf on* Bun_G*), whose Hecke eigenvalue is isomorphic to* $\mathrm{Hyp}_{\check{G}}(\underline{\lambda})$.

(ii) *There exists a Frobenius structure on* $\mathrm{Hyp}_{\check{G}}(\underline{\lambda})$, *whose Frobenius trace are certain hypergeometric sums.*

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- Type A case is due to Kamgarpour–L. Yi.
- Beilinson–Drinfeld's approach for geometric Langlands.
- $\mathrm{Hyp}_{\check{G}}(\underline{\lambda})$ satisfy certain functorial relationship for $\mathrm{SO}_{2r+1} \to \mathrm{SL}_{2r+1}$, $\mathrm{Sp}_{2r} \to \mathrm{SL}_{2r}$, $\mathrm{SO}_{2r+1} \to \mathrm{SO}_{2r+2}$, generalizing that of Kloosterman sheaves for reductive groups.

Thank You!